Optimal Frequency Ratio for Chords
和音に最適な周波数比

RITSUMEIKAN

立命館高等学校　今泉凌

Abstract

This research aims to determine the optimal frequency ratios to enhance chord quality. While equal temperament, a commonly used tuning method for instruments like the piano, has its merits, it cannot produce the best-sounding chords. To address this issue, adjusting the intervals is essential to obtain better-sounding chords. The beauty of chords is closely linked to the least common multiple (LCM) of their frequency ratios. Through this research, optimal frequency ratios have been identified for various chord types: Major chords: 4:5:6, Minor chords: 10:12:15, Augmented chords: 16:20:25, Diminished chords: 5:6:7, and Suspended 4th chords: 6:8:9. Utilizing these ratios helps musicians create more beautiful chords. For future research, an instrument will be created to be used on a computer or mobile phone. This instrument will be able to adjust optimal intervals automatically.

Introduction

Chord

A chord is a composite sound when three or more notes of different pitches are played at the same time. It produces harmony and the harmony is said to sound better when its frequency ratio is expressed as a simple integer ratio.

Equal Temperament

Equal temperament, which is widely used today, is one octave divided into 12 parts where the ratios between adjacent notes are equal. Since the intervals are equal, it is possible to play in different keys. However, it does not produce a beautiful chord sound because the frequency ratios of the chords cannot be expressed in integer ratios.

Just Intonation

Just intonation is an excellent temperament for making beautiful sounding chords. In just intonation, the C major chord (C-E-G) sounds beautiful. This is because the frequency ratio is 4:5:6, which is a simple integer ratio. However, D minor chord (D-F-A) does not sound good. This is because the frequency ratio is 27:32:40, which is not a simple integer ratio. It is considered that by adjusting the intervals of the D minor chord, more beautiful chords can be played.

Beauty of Chords

Chords with smaller LCM of frequency ratios could be considered to sound more beautiful because of the shorter period of the composite waveforms. In just intonation, the frequency ratio of the C major chord (C-E-G) is 4:5:6, so its LCM is 60; the frequency ratio of the D minor chord (D-F-A) is 27:32:40, so its LCM is 4320. Therefore, the C major chord with the LCM of 60 is considered more beautiful than the D minor chord with the LCM of 4320.

Experiment 1

Aim

To determine the optimal frequency ratio of basic triads.

Materials

Basic triads on the right were used for Experiment 1:

- Major chord
- Minor chord
- Augmented chord
- Diminished chord
- Suspended 4th chord

Step 1: Sort Frequency Ratios

1. The frequency ratios of the triads into Table 1 was set so that the largest value was less than twice the smallest value within the same octave.
2. The order of LCM was sorted from the smallest to the largest.
3. Only ratios with the greatest common divisor (GCD) of 1 was listed.

Table 1 indicates the triads that fit in one octave which sound better toward the top.

Table 1 Optimal frequency ratios of triads

Frequency ratio			LCM	GCD
3	4	5	60	1
4	5	6	60	1
10	12	15	60	1
12	15	20	60	1
6	8	9	72	1
8	9	12	72	1

Step 2: Examine Notes from Ratios

The frequency ratio of basic chords was examined to determine which notes were used for the chord based on the frequency ratio that was obtained in Step 1.

For example, the top row of Table 2 shows that a chord with the frequency ratio of 3:4:5 is composed of the notes C-F-A.

Table 2 Relation between a frequency ratio and notes

C	C# Db	D	D# Eb	E	F	F# Gb	G	G# Ab	A	A# Bb	B
3					4				5		
4				5			6				
10			12				15				
12				15					20		
6					8		9				
8		9					12				

Step 3: List Components of Chords

The component notes from basic chords was confirmed.

Table 3 lists the notes compose the basic chords: Major (maj), Minor (min), Augmented (aug), Diminished (dim), and Suspended 4th (sus4).

Table 3 List of chord types and its component notes

Chord type	Component notes		
maj	C	E	G
min	C	Eb	G
aug	C	E	G#
dim	C	Eb	Gb
sus4	C	F	G

Step 4: Determine Optimal Ratios

The optimal frequency ratio for each chord was determined by considering Step 2 and Step 3.

Table 4 lists the optimal frequency ratios for basic chords: Major: 4:5:6, Minor: 10:12:15, Augmented: 16:20:25, Diminished: 5:6:7, and Suspended 4th: 6:8:9.

Table 4 Optimal frequency ratios for each chord

Chord type	Ratio	LCM
maj	4 : 5 : 6	60
min	10 : 12 : 15	60
aug	16 : 20 : 25	400
dim	5 : 6 : 7	210
sus4	6 : 8 : 9	72

Experiment 2

Aim

To utilize the optimal frequency ratio for basic chords determined in Experiment 1, the intervals were adjusted to improve the sound of chords by using just intonation.

The ratio of the intervals in just intonation is as follows:

Table 5 Just intonation intervals

Notes	C	D	E	F	G	A	B
Ratio to C	1/1	9/8	5/4	4/3	3/2	5/3	15/8

Step 1: Diatonic Chords in C Major Scale

The diatonic chords in the C major scale are the following seven chords shown in Table 6 such as C major, D minor, E minor, F major, G minor, A minor, B diminished. Since these are the most important chords, these chords had to be able to be played at the optimal frequency ratio.

Table 6 Diatonic chords and frequency ratios in just intonation

Diatonic chord	Component notes	Chord type	Actual ratio	Optimal ratio
C major	C-E-G	maj	4:5:6	4:5:6
D minor	D-F-A	min	27:32:40	10:12:15
E minor	E-G-B	min	10:12:15	10:12:15
F major	F-A-C	maj	4:5:6	4:5:6
G major	G-B-D	maj	4:5:6	4:5:6
A minor	A-C-E	min	10:12:15	10:12:15
B diminished	B-D-F	dim	45:54:64	5:6:7

Table 6 shows that two chords, D minor and B diminished, are not optimized. Because the actual frequency ratio does not match the optimal frequency ratio.

Step 2: Adjust Intervals

Table 7 below lists the notes that compose each diatonic chord and their intervals. For D minor chord and B diminished chord, the original intervals were adjusted to obtain the optimal frequency ratio.

Table 7 Notes that compose diatonic chords

	C	D	E	F	G	A	B
C major	1/1		5/4		3/2		
D minor		**10/9**		4/3		5/3	
E minor			5/4		3/2		15/8
F major	1/1			4/3		5/3	
G major		9/8			3/2		15/8
A minor	1/1		5/4			5/3	
B diminished		9/8		**21/16**			15/8

When playing the D minor chord, the D note was adjusted from its original ratio of 9/8 to 10/9 to obtain the optimal frequency ratio of 10:12:15; when playing the B diminished chord, the F note was adjusted from 4/3 to 21/16 to get 5:6:7.

Conclusions / Future Plan

Conclusions

The optimal frequency ratio of basic chords was determined in terms of the least common multiple. This allowed me to adjust the intervals of the D minor chord and B diminished chord, which did not sound very good in just intonation, to the optimal frequency ratio.

Future Plan

The following two things are planned:

1. To find a more accurate method to calculate the frequency ratio because when the least common multiple of the frequency ratio is small but the number of the ratio is large, it could not be processed.
2. To create an instrument that automatically adjusts to the optimal interval when a chord is played, since it is difficult to use the optimal interval depending on the situation.

References

1. 桜井進、坂口博樹(2011)音楽と数学の交差
2. 坂口博樹(2016)数と音楽
3. ピタゴラス音律から新しい音律へ、馬場良始、URL : http://www.osaka-kyoiku.ac.jp/~ybaba/onritsu2.pdf
4. Realtime Procedural Audio and Synthesized Piano in Unity 3D, Shahriar Shahrabi, URL : https://shahriyarshahrabi.medium.com/realtime-procedural-audio-and-synthesized-piano-in-unity-3d-a98496a6df5e
5. 中村 健太郎(2010)「図解雑学 音のしくみ」ナツメ社